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Scaling properties of trough densities in sandpile models

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We characterize the self-organized criticality of a class of one-dimensional sandpile models (limited local models) in terms of the scaling properties of troughs whose spatial distribution controls the size of avalanches. We establish, both by a mean-field approximation and simulations, that the trough density ρ_k is exponentially small in the depth k and that the algebraic decay of the local, overall trough density is governed by the diffusion singularity in the continuum limit studied by Carlson *et al.* [Phys. Rev. Lett. **65**, 2547 (1990)]. Multiscaling is seen to arise from the correlations neglected in the approximation.

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Bak, Tang, and Wiesenfeld [1] proposed a few years ago a far-reaching concept termed self-organized criticality (SOC) as the underlying mechanism that generates both spatial and temporal scale invariance in dissipative dynamical systems in nature. One of its appealing features is that no tuning is necessary to reach scale invariant states.

Because of their ease of implementation, cellular-automata sandpile models are usually used to advocate SOC. One particular model that has caught much recent attention [2–9] is the so-called "local limited" (L^2) model introduced by Kadanoff *et al.* [2]. This model is especially simple in one dimension and is defined by a threshold dynamics of the slope variable $z(x)$, $x=1,\ldots,L$, and the boundary conditions. During each time step, the addition of a grain at a randomly chosen column x updates $z(x)$ by $+1$, and $z(x-1)$ by -1. If $z(x)>z_0$ (the threshold), the column becomes unstable and N grains slide down to the right onto column $x+1$. Subsequent local instabilities may be triggered, until the pile settles into a globally stable configuration. Then another grain is added to initiate the next time step. We will confine our attention to the simplest case, $z_0=2=N$. As is common in nonequilibrium systems, the boundary conditions are important: only in the case of open boundaries (e.g., closed at $x=0$ and open at $x=L+1$, hereafter called OBC) will the system evolve on its own into an SOC state by regulating the flow at the open end. By contrast, z is conserved under periodic boundaries (PBC's) and therefore must be tuned to attain criticality [3].

Carlson *et al.* [3] observed that grains sliding down the slope in the L^2 model are trapped at $z(x)\le z_0-N=0$. They called those sites *troughs*. The essence of the trough terminology is that *all* physical events occurring on the pile can be classified according to whether troughs are created, lifted, shifted, or coalesced. The SOC state is characterized by avalanches that cover a wide range of spatial extent. Its distribution can be expressed in terms of that of troughs [4,6]. For this reason, the scaling properties and correlations of troughs, which are the subject of this article, are important characteristics of the associated SOC state.

For convenience, we label each site either as marginal ($z=2$), stable ($z=1$), or a trough of depth $k=-z\in\{0,1,\ldots,k_{\max}\}$, where $k_{\max}=L-x-1$ at site x in the L^2 model with OBC [7]. Let p be the fraction of stable sites among $z>0$ ones, and $\rho_k=L^{-1}\langle\sum_x\delta_{z(x),-k}\rangle$ the mean density of trough of depth k. The total mean density of troughs is $\rho=\sum_k\rho_k$, and the density of stable sites is $\rho_s=(1-\rho)p$ [that of marginal sites is $\rho_m=(1-\rho)(1-p)$].

For a grain dropped on x, label the affected pair by $\{z(x-1)z(x)\}$. The event that triggers avalanches is described by the change [3]

$$\{z(x_L),\{22\},z(x_T),z(x_R)\}$$
$$\rightarrow\{z(x_L)+2,\{11\},z(x_T)-2,z(x_R)+2\}\quad\text{coalescence},$$

where x_L and x_R are the positions of the first trough to the left and right of x, and the mass ($M=x-x_L$) that has slid down extends from the tail at $x_T=x_L+x_R-x$ to the front at x_R. Similarly,

$$\{\{12\},z(x_R-1),z(x_R)\}$$
$$\rightarrow\{\{21\},z(x_R-1)-2,z(x_R)+2\}\quad\text{slide I};$$
$$\{\{z_L 2\},z(x_R-1),z(x_R)\}$$
$$\rightarrow\{\{z_L+1,1\},z(x_R-1)-2,z(x_R)+2\}\quad\text{slide II}.$$

All other events are local, two-site processes, such as $\{20\}\rightarrow\{11\}$.

It is the nonlocal, multisite processes (coalescence and slides) that dominate the dynamics of sandpile models in general, and are responsible for their nontrivial scaling behavior. These processes involve joint probabilities such as the five-point probability for the coalescence, $P_5(z_L,\{22\},z_T,z_R;x_L,x,x_R)$, which is the probability of finding the pair $\{22\}$ at x, its closest troughs z_L and z_R at x_L ($<x$) and x_R ($>x$), and a nontrough z_T at $x_T=x_L+x_R-x$, and $P_4(\{12\},z(x_R-1),z(x_R);x,x_R)$ for slide I. Apparently, these probabilities carry much more information than is needed, and are not easy to determine. In the equations of motion for the mean densities, we only need

$$\bar{P}_5(z_L,\{22\},z_T,z_R) \equiv L^{-1}\sum_x\sum_{x_L}\sum_{x_R} P_5(z_L,\{22\},z_T,z_R;x_L,x,x_R) , \quad (1)$$

and likewise for P_4. The factor $L^{-1}\sum_x$ comes from random seeding. Moreover, it is easy to see that the avalanche distribution

$$P(M) \propto \sum_{z_L\,(\le 0)} P_2(z_L,2;x_L,x=x_L+M) ,$$

and that the cluster-size (intertrough distance) distribution

$$\tilde{P}(s) \propto \sum_{z_L\,(\le 0)}\sum_{Z_R\,(\le 0)} P_2(z_L,z_R;x_L,x_R=x_L+s) .$$

Unfortunately, even these simpler probabilities are nontrivial and are not available analytically [9,6]. The difficulty arises from the correlations among *all* the sites between the two end points.

It is possible to write the equations of motion for ρ_m, ρ_s, and ρ_k in terms of $\bar{P}_n$'s. For instance, denoting $\rho_m(t+1)-\rho_m(t)$ by $d\rho_m/dt$, we find

$$\begin{aligned} L\frac{d\rho_m}{dt} =& \sum_{z_R\,(\ne 0)} \bar{P}_4(\{12\},2,z_R)+\bar{P}_4(\{12\},1,0)-2\sum_{z_L\,(\ne 0)}\sum_{z_R\,(\ne 0)} \bar{P}_5(z_L,\{22\},1,z_R) \\ &-\sum_{z_L\,(\ne 0)} \bar{P}_5(z_L,\{22\},1,0)-3\sum_{z_L\,(\ne 0)}\sum_{z_R\,(\ne 0)} \bar{P}_5(z_L,\{22\},2,z_R)-2\sum_{z_L\,(\ne 0)} \bar{P}_5(z_L,\{22\},2,0) \\ &-\sum_{z_R\,(\ne 0)} \bar{P}_5(0,\{22\},1,z_R)-2\sum_{z_R\,(\ne 0)} \bar{P}_5(0,\{22\},2,z_R)-\bar{P}_5(0,\{22\},2,0) \\ &-\sum_{z\,(\le 0)}\sum_{z_R\,(\ne 0)} \bar{P}_4(\{z2\},1,z_R)-2\sum_{z\,(\le 0)}\sum_{z_R\,(\ne 0)} \bar{P}_4(\{z2\},2,z_R)-\sum_{z\,(\le 0)} \bar{P}_4(\{z2\},2,0)+(\text{two-site terms}). \end{aligned} \quad (2)$$

Its derivation is straightforward. Consider the slide-I event as an example: by definition of z_R, $z(x_R-1)=1$ or 2. The process $\{\{12\},1,z_R\}\rightarrow\{\{21\},-1,z_R+2\}$ increases the number of marginal sites by 1 if $z_R=0$, and the process $\{\{12\},2,z_R\}\rightarrow\{\{21\},0,z_R+2\}$ decreases that number by 1 if $z_R\le -1$. This accounts for the first two terms. Likewise, coalescence contributes to the $\bar{P}_5$ pieces, and slide II contributes to the last three $\bar{P}_4$ terms. The equations for other densities can be derived in the same manner. The complexity of multisite correlations does not allow more explicit rendering.

We now turn to a mean-field-type approximation. The motivation is that scaling behavior is dominated by large-scale events, since the mean intertrough distance goes up in L very rapidly ($\sim L^{1/3}$) [3,8]. That means, e.g., in a *typical* coalescence event, $x-x_L$ and x_R-x diverge with L. Since the joint probabilities factorize into products in the limit of infinite distances, we invoke a decoupling scheme as a first approximation; e.g.,

$$\bar{P}_5(z_L,\{22\},z_T,z_R)\sim P_1(z_L)P_1^2(2)P_1(z_T)P_1(z_R) , \quad (3)$$

where P_1's are singlet probabilities expressible in terms of p and ρ_k's. Thus, for instance, the first term in Eq. (2) factorizes into $-(1-\rho)^2p(1-p)(1-p)(1-\rho_0/\rho)$. Whether the scaling properties of the model are robust enough to hold up to such an approximation will be determined by numerical tests.

Under this scheme, it is tedious but straightforward to derive the equations for the mean densities by taking *all* processes into account. Defining $q_k=\rho_k/\rho$, we find

$$\begin{aligned} L\frac{d\rho_s}{dt} =& p^2-5p+2+(p^2-3p+2)q_1-(p^2-5p+2)\rho \\ &+(2-p)\rho_0-(2p^2-5p+3)\rho_1-(1-p)\rho\rho_0 \\ &+(1-p)^2\rho\rho_1 , \end{aligned} \quad (4)$$

$$\begin{aligned} L\frac{d\rho_0}{dt} =& 2p^2-2p+1+(p^2-3p+2)(q_2-q_0) \\ &-(3p^2-3p+1)\rho+(2p^2-5p+1)\rho_0 \\ &+(2-p)\rho_1-(2p^2-5p+3)\rho_2+(p^2-p)\rho^2 \\ &-(1-p)^2\rho\rho_0-(1-p)\rho\rho_1+(1-p)^2\rho\rho_2 , \end{aligned} \quad (5)$$

$$\begin{aligned} L\frac{d\rho_1}{dt} =& (p-p^2)+(p^2-3p+2)(q_3-q_1)+(p^2-p)\rho \\ &+(2p^2-5p+1)\rho_1+p\rho_0+(2-p)\rho_2 \\ &-(2p^2-5p+3)\rho_3+(1-p)\rho\rho_0 \\ &-(1-p)^2\rho\rho_1-(1-p)\rho\rho_2+(1-p)^2\rho\rho_3 , \end{aligned} \quad (6)$$

$$\begin{aligned} L\frac{d\rho_k}{dt} =& (p^2-3p+2)(q_{k+2}-q_k)+p\rho_{k-1} \\ &+(2p^2-5p+1)\rho_k+(2-p)\rho_{k+1} \\ &-(2p^2-5p+3)\rho_{k+2}+(1-p)\rho\rho_{k-1} \\ &-(1-p)^2\rho\rho_k-(1-p)\rho\rho_{k+1}+(1-p)^2\rho\rho_{k+2} \\ &\text{for } k\ge 2 . \end{aligned} \quad (7)$$

Notice that the derivation makes no explicit reference to the boundary conditions, as is common in mean-field approximations. The SOC steady state, characterized by the variables p and ρ_k ($k \geq 0$), can be obtained by setting all right-hand sides equal to zero. For large, finite L and near the critical point, the system has a small parameter $\rho \ll 1$. This motivates us to pursue a power-series solution in ρ:

$$p=\sum_n a_n\rho^n,\quad q_0=\sum_n b_n\rho^n,\quad q_1=\sum_n c_n\rho^n\ ,$$
$$q_2=\sum_n d_n\rho^n, \ldots .$$

It is known that the height variable, $h(x)=\sum_{x'=x}^{L} z(x')$, obeys $y+1 \leq h(L-y) \leq 2(y+1)$ under OBC [7], which yields $1-y \leq z(L-y) \leq 2$. Although the relative weight of these local states is unknown, the rarity of deep troughs is clear from these bounds—maximum depth $[z(L-y)=1-y]$ occurs, provided $h(L-y')=2(y'+1)$ for *all* $y'<y$. Precisely how rare is what we now determine. Using (7), we get

$$L \sum_{j=k,k+2,k+4,\ldots} d\rho_j/dt$$
$$=-(p^2-3p+2)q_k+p\rho_{k-1}+O(\rho_k) \quad \text{for } k \geq 2\ ,$$

hence $\rho_{k-1} \sim q_k \gg \rho_k$ follows. Plugging the power series into (4)–(7), one finds order by order that

$$p=\tfrac{1}{2}+\tfrac{1}{6}\rho+O(\rho^2)\ , \tag{8}$$
$$q_0=\tfrac{2}{3}-\tfrac{1}{27}\rho+O(\rho^2)\ , \tag{9}$$
$$q_1=\tfrac{1}{3}-\tfrac{5}{27}\rho+O(\rho^2)\ , \tag{10}$$
$$q_k=\tfrac{1}{2}\left[\tfrac{2}{3}\right]^k\rho^{k-1}+O(\rho^k) \quad \text{for } k \geq 2\ . \tag{11}$$

Obviously the series can be improved systematically. The finite-size effect implied by (8) has been noted by Krug [4]. The first terms of q_0 and q_1 have an important implication: while it is true that the trough *density* becomes extinct in the thermodynamic limit [3], the ratios converge to finite values. Therefore, the SOC state remains well defined by the scaling behavior of the distribution of event sizes. These predictions agree well with simulations (see Fig. 1), although b_0, c_0, and the amplitudes expectedly deviate from the measured ones. The exponential drop of trough densities in depth is verified up to $k=3$, while data for deeper ones are too noisy to be useful.

$a_0=\frac{1}{2}$ can also be obtained by parity relations—apparently our approximation respects such a symmetry. Let $\sigma(x)=(-1)^{z(x)}$ be the parity variable. The fraction of sites being even and odd is

$$f_+=\frac{1}{2L}\left\langle\sum_x[1+\sigma(x)]\right\rangle=(1-\rho)(1-p)+\rho_0+\rho_2+\cdots\ ,$$
$$f_-=\frac{1}{2L}\left\langle\sum_x[1-\sigma(x)]\right\rangle=(1-\rho)p+\rho_1+\rho_3+\cdots\ . \tag{12}$$

Random seeding implies $\langle\sigma(x)\rangle=0$ [4,8], so $f_+=f_-=\frac{1}{2}$ for any L in steady state (they relax in time

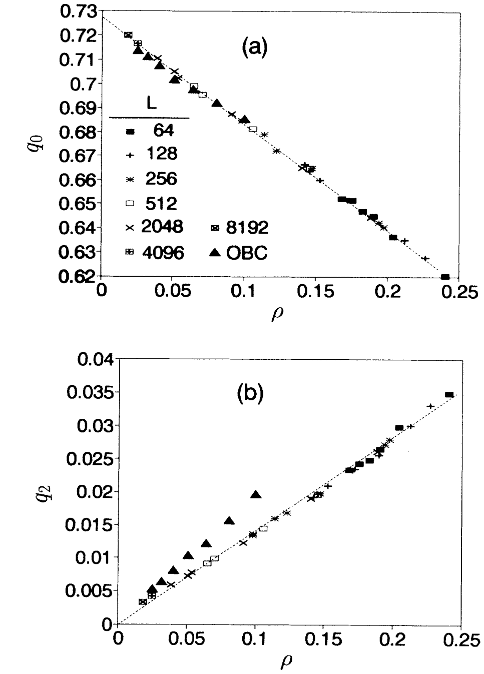

FIG. 1. Fractional trough densities. Lines are fits to PBC data for (a) $q_0 \approx 0.726-0.42\rho$, and (b) $q_2 \approx 0.136\rho$. Also shown is the convergence of OBC ($L=64$ to 4096) to the same critical point as $L \to \infty$. Not shown due to lack of space are $p \approx \frac{1}{2}+0.20\rho$ and $q_1 \approx 0.274+0.28\rho$.

as $e^{-2t/L}$). Using the series expansions and (12), we readily deduce that

$$a_0=\tfrac{1}{2},\quad a_1=b_0-\tfrac{1}{2}=\tfrac{1}{2}-c_0,\quad c_1=a_1-a_2=-(b_1+d_1)\ . \tag{13}$$

Hence the average slope is

$$\bar{z}=\frac{1}{L}\sum_x z(x)$$
$$=2(1-\rho)(1-p)+(1-\rho)p-\sum_{k(\geq 1)} k\rho_k$$
$$=z_c-2\rho-2d_1\rho^2+O(\rho^3)\ , \tag{14}$$

which is confirmed nicely to this order by simulations (see Fig. 2). Under PBC, this prescribes the way the trough density has to vanish [3] as we tune the conserved average slope $\bar{z} \nearrow z_c=\frac{3}{2}$.

For OBC, we find numerically that (8)–(14) also apply to the corresponding *local* quantities. This is conceivable by virtue of scale invariance manifested by the algebraic approach of the coarse-grained slope $z(x)$ to z_c via $x^{-1/(\phi-1)}$, where $\phi(=4)$ characterizes the singularity of the diffusivity [3]. Scale invariance suggests that the *composition* of the troughs in a small local region is the same as that of the means. Thus, e.g., one might expect $\rho_0(x) \approx q_0\rho(x)$ and $\rho_k(x) \propto \rho(x)^k$. This is indeed numerically the case. From the generalization of (14) to local $z(x)$ and $\rho(x)$, we infer that $\rho(x) \approx Ax^{-1/(\phi-1)}$, which

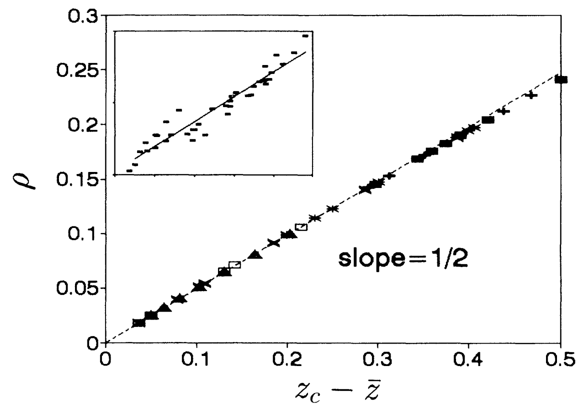

FIG. 2. Confirmation of Eq. (14) for PBC. Same symbols as Fig. 1. Inset plots $(z_c-\bar{z}-2\rho)/2\rho$ vs ρ, with slope $=0.14\approx\rho_2/\rho^2\approx d_1$.

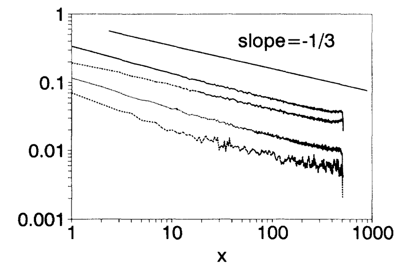

FIG. 3. Algebraic decays of $\rho(x)$, $\rho_0(x)$, $\rho_1(x)$, and $\rho_2(x)/\rho(x)$ from top to bottom, for OBC, $L=512$. The slope $-\frac{1}{3}$ is related to the singularity in diffusion equations (see text).

provides a direct way to determine ϕ (see Fig. 3). Numerically, we find that A is independent of L, thus consistent with $\bar{\rho}\sim L^{-1/3}$ [3,8] upon integration over x.

The decoupled approximation presented above is reminiscent of the free ("noninteracting") theory in the field-theoretic formulation of critical phenomena [10]. It is interesting to see if an analogous perturbation theory can be developed. To see which direction it would lead us, note that multiscaling is absent under decoupled approximation; e.g., in the cluster size distribution, we find $\langle s^n\rangle \propto \langle s\rangle^n \sim \rho^{-n}$. Undoubtedly multiscaling results from the neglected fluctuations. To decide what correlation is important, we seek the causes of the $O(1)$ deviations in the estimates of q_0 and q_1. We find that [11] the right sides of $Ld\rho_s/dt$ and $Ld\rho_m/dt$ are both $\sim\text{const}+O(\rho)$, but $Ld\rho/dt=-Ld(\rho_s+\rho_m)/dt<O(\rho^2)$. The origin of the differences can be traced to z_T: it appears in the former pairs, but not in $d\rho/dt$. Hence the nontrough z_T is strongly (long-range) correlated with the other relevant sites in the events, incurring $O(1)$ errors when replaced by singlet probabilities.

In conclusion, despite the crudeness of our approximation, it captures correctly certain scaling properties of the model, and may serve as a starting point for a more sophisticated theory for the correlations.

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[1] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987); P. Bak and K. Chen, Sci. Am. **264** (1), 26 (1991).
[2] L. P. Kadanoff, S. R. Nagel, L. Wu, and S.-m. Zhou, Phys. Rev. A **39**, 6524 (1989).
[3] J. M. Carlson, J. T. Chayes, E. R. Grannan, and G. H. Swindle, Phys. Rev. A **42**, 2467 (1990); Phys. Rev. Lett. **65**, 2547 (1990).
[4] J. Krug, J. Stat. Phys. **66**, 1635 (1992).
[5] J. Krug, J. E. S. Socolar, and G. Grinstein, Phys. Rev. A **46**, R4479 (1992).
[6] K.-t. Leung, Phys. Rev. A **46**, 6211 (1992).
[7] S.-C. Lee and W.-J. Tzeng, Phys. Rev. A **45**, 1253 (1992).
[8] L. P. Kadanoff, A. B. Chhabra, A. J. Kolan, M. J. Feigenbaum, and I. Procaccia, Phys. Rev. A **45**, 6095 (1992).
[9] A. B. Chhabra, M. J. Feigenbaum, L. P. Kadanoff, A. J. Kolan, and I. Procaccia, Phys. Rev. E **47**, 3099 (1993).
[10] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (Clarendon, Oxford, 1989).
[11] To isolate such causes, we improve the approximation by pairwise decoupling, incorporating nearest-neighbor correlations into the equations, and having them checked numerically. For instance, slide events have a term $\propto[P_2(\{12\})[P_2(10)-\sum_{z_R(\le -1)}P_2(2z_R)]$ in $d\rho_s/dt$.